Lecture 10: Hypothesis Testing

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Hypothesis testing

- Hypothesis: A statement about a population such that:
  - It is either true or false (but not both)
  - With full knowledge of the population data, it is possible to identify, with certainty, whether it is true or false.
- Examples:
  - The mean daily income of a family in an undeveloped country is less than $1 U.S.
  - The average number of elderly people in the U.S. who get a flu during a winter is 1,000,000
  - The average earnings of NASDAQ companies during the last decade is $500 million dollars.

Hypothesis testing

- Due to sample error, it is impossible to make correct decision about a hypothesis all the time.
- The objective instead is:
  - Design a process such that the probability of making a wrong decision is as small as possible.
  - Particularly, in hypothesis testing, the objective is to select sample data from the population and, based on analysis of this sample data, decide whether a hypothesis is true or false with a given level of significance.

An illustrative example

Court Decision / Reality	Person is declared 'not guilty'	Person is declared "guilty"
Person is "innocent" H_0 is true	Correct Decision	Error Type I Error
Person is "guilty" H_1 is true	Error Type II Error	Correct Decision

H_0: person is innocent H_1: person is guilty

An illustrative example

- **Alternative Hypothesis**: H_1: The hypothesis that we are interested in proving.
- **Null hypothesis**: H_0: The complement of the alternative hypothesis.
- **Type I error**: reject the null hypothesis when it is correct.
  - It is measured by the level of significance α, i.e.,
  - the probability of type I error. This is the probability of falsely rejecting the null hypothesis.
- **Type II error**: do not reject the null hypothesis when it is wrong.
  - It is measure by the probability of type II error β.
  - Furthermore we call $1 - \beta$ to be the **power of test**, which is the probability of correctly rejecting the null hypothesis.
- **Critical value**: The dividing point between the region where the null hypothesis is rejected and the region where it is not rejected

Hypothesis testing

- Given a pair of complementary hypothesis about a population, called null (H_0) and alternative (H_1) hypothesis, hypothesis testing is the technique of analyzing sample data to make one of the following two decisions:
  - We have enough evidence to reject Ho in favor of H_1.
  - We don't have enough evidence to reject Ho in favor of H_1.
- The test is designed to keep the probability of Type I error equal to α, and minimize the probability of Type II error β.

Layout

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2. Five step procedure for testing a hypothesis
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Five step procedure for testing a hypothesis

- State the null and alternate hypotheses
- Select the level of significance
- Identify the test statistic
- State the decision rule
- Compute the value of the test statistic and make a decision:
    - There is enough evident to reject H_0 in favor of H_1;
    - There is not enough evident to reject H_0 in favor of H_1.

Five step procedure for testing population mean

- Step 1. We distinguish two kinds of HT, which decides the decision rule used in Step 4:
  - Two-tailed HT: Value assumed under H_1 is on either side of the values assumed under H_0.
  - Example:

$$\begin{aligned} H_0: &\quad \mu = \mu_0 \\ H_1: &\quad \mu \neq \mu_0 \end{aligned}$$

  - One-tailed HT: Every value assumed under H_1 is on one side of every value assumed under H_0.
  - Example:

$$\begin{aligned} H_0: &\quad \mu \leq \mu_0 \\ H_1: &\quad \mu > \mu_0 \end{aligned}$$

Five step procedure for testing population mean

- Step 2. The most often used level of significance are:

α	z_α
0.05	1.65
0.025	1.96
0.01	2.33
0.005	2.58

Five step procedure for testing population mean

- Step 3. Test Statistic to be used: Based on the assumptions:
  - If $n \geq 30$ and σ is known; Or, if $n < 30$, σ is known, and the population is normally distributed.

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

  - If $n \geq 30$ and σ is unknown, the standard deviation s of the sample is used to approximate the population standard deviation σ:

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

  - If $n < 30$, σ is unknown, and the population is normally distributed, then we should use the Student t distribution:

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

- Step 4. Decision rule to be used:

$$\text{If} \quad |z| = \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| \geq \begin{cases} z_{\alpha/2}, & \text{two-tailed;} \\ z_{\alpha}, & \text{one-tailed;} \end{cases}$$

$$\text{then} \quad \text{reject } H_0$$

Five step procedure for testing population mean

- Step 5. Calculate the statistic specified at Step 3 and Make your decision base don the rule in Step 4.

Example

- Example: The mean annual turnover rate of the 200-count bottle of Bayer Aspirin has in the past been 6.0 with a standard deviation of 0.50. (This indicates that the stock of Bayer Aspirin turns over on the pharmacy shelves an average of 6 times per year). A random sample of 64 bottles of the 200-count size Bayer Aspirin showed a mean turnover rate of 5.84 times per year. It is suspected that the mean turnover rate has changed and is not 6.0. We wish to test if there is sufficient evidence, at 5% significance level, to conclude that the mean turnover rate has changed. Assume that the population standard deviation has not changed.

Five step procedure for testing population mean

- Step 1. We have a two-tailed HT here

$$
\begin{aligned}
H_0: &\quad \mu = 6.0 \\
H_1: &\quad \mu \neq 6.0
\end{aligned}
$$

- Step 2. The significance level is $\alpha = 0.05$ and $z_{\alpha/2} = z_{0.025} = 1.96$.
- Step 3. Since $n = 64$ and $\sigma = 0.5$ is known, so we use

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

- Step 4. Since it is two-tailed test, we use the decision rule:

$$\text{If } |z| = \left| \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \right| \geq z_{\alpha/2}, \text{ reject } H_0$$

- Step 5. we now calculate the statistic specified at Step 3 and Make your decision base don the rule in Step 4.

$$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{5.84 - 6.0}{0.5/\sqrt{64}} = -2.56.$$

  - Since $|-2.56| > 1.96$, we reject H_0 in favor of H_1, concluding that the mean turnover rate has indeed changed and is not 6.0 anymore at 0.05 level of significance.

Example

- Example: The manufacturer of the X-15 steel-belted radial truck tire claims that the mean mileage the tire can be driven before the thread wears out is at least 80,000 km. The standard deviation of the mileage is 8,000 km. The Crosset Truck Co. bought 48 tires and found that the mean mileage for their trucks is 79,200 km. From Crosset's experience, can we conclude, with 5% significance level, that the manufacturer's claim is wrong?

Five step procedure for testing population mean

- Step 1. We have a one-tailed HT here

$$\begin{aligned} H_0: &\quad \mu \geq 80,000 \\ H_1: &\quad \mu < 80,000 \end{aligned}$$

- Step 2. The significance level is $\alpha = 0.05$ and $z_\alpha = z_{0.05} = 1.65$.
- Step 3. Since $n = 48$ and $\sigma = 8,000$ is known, so we use

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

- Step 4. Since it is one-tailed test, we use the decision rule:

$$\text{If } |z| = \left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq z_\alpha, \text{ reject } H_0$$

- Step 5. we now calculate the statistic specified at Step 3 and Make your decision base don the rule in Step 4.

$$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{79,200 - 80,000}{8,000/\sqrt{48}} = -0.693.$$

  - Since $|-0.693| < 1.65$, we do not have enough evidence to reject H_0 in favor of H_1; that is, we don't have enough evidence to claim that the manufacturer is incorrect. at 0.05 level of significance.

p-value

- We use z-value (z-score) or t-value in the decision rule (Step 4).
- Equivalently we can also use p-value:
  - A p-value is the probability, (assuming that the null hypothesis is true) of finding a value of the test statistic at least as extreme as the computed value for the test!

$$p\text{-value} = \begin{cases} 2P\left(Z \geq \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right|\right), & \text{two-tailed;} \\ P\left(Z \geq \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right|\right), & \text{one-tailed.} \end{cases}$$

Decision Rule using p-value (Step 4)

- Reject H_0 whenever the p-value is no more than the level of significance α (that is p-value $\leq \alpha$).

Example

- **Example**: Reconsider the tire manufacturer example. It is one-tailed and the p-value can be calculated as:

$$P\left(Z \geq \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right|\right) = P\left(Z \geq \left|\frac{79{,}2000-80{,}000}{8{,}000/\sqrt{48}}\right|\right) = P(Z \geq |-0.693|) = 0.5 - 0.2549 = 0.2451$$

  - Since p-value is larger than 0.05, do not reject H_0.

- **Example**: Reconsider the Bayer Aspirin turnover rate example. It is two-tailed and the p-value can be calculated as:

$$2P\left(Z \geq \left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right|\right) = 2P\left(Z \geq \left|\frac{5.84-6.0}{0.5/\sqrt{64}}\right|\right) = 2P(Z \geq |-2.56|) = 2(0.5 - 0.4948) = 0.0104$$

  - Since p-value is smaller than 0.05, reject H_0.